SQUIDS

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Abstract

Superconducting Quantum Interference Devices (SQUIDs) are highly sensitive magnetometers that leverage the quantum properties of Josephson junctions to measure small changes in magnetic fields. By analyzing Cooper pair tunneling and quantum interference in Josephson junctions under an applied magnetic field, we derive the key equations of SQUIDs. We then show that a typical SQUID can achieve a magnetic field sensitivity on the order of 10 picotesla.

1 Introduction

SQUIDs exploit principles of superconductivity and quantum mechanics to achieve remarkable precision in measurement of magnetic fields. Since first developed in the 1960s, SQUIDs have progressed to measure fields as small as a few femtoteslas. In this paper, we derive and use the properties of a Josephson Junction to construct a simple SQUID and show its theoretical sensitivity to be about 10 picotesla. Notably, we will constrain the discussion to the mathematically simpler, low-temperature superconductors.

2 Background

2.1 Wavefunction of Superconductors

In Ginzburg-Landau theory of superconductivity, the switch from normally conducting to superconducting is a second order phase transition [1]. Such a transition can be represented by a free energy functional. This is an expression of the free energy as a function (primarily) of temperature which is Taylor expanded in a complex order parameter around the critical temperature. The order parameter is taken to be a complex wavefunction Ψ describing the density of Cooper pairs:

$$|\Psi|^2 = n \tag{1}$$

Where n is the density of Cooper pairs. In this formulation the formation of Cooper pairs is due to the minimization of the free energy functional. The formation process happens gradually as T crosses T_c, the critical temperature of superconductivity. Therefore, near T_c, the density of Cooper pairs gradually increases to 1 as the material undergoes the phase transition. We can also express the order parameter as a magnitude and phase:

$$\begin{aligned} |\Psi| e^{-i\phi} &= \sqrt{n} \\ |\Psi| &= \sqrt{n} \cdot e^{i\phi} \end{aligned} \tag{2}$$

Where ϕ is the phase of the wavefunction Ψ.

2.2 Gauge Invariance

In quantum mechanics, the Hamiltonian for a particle in an electromagnetic field can be expressed in terms of the vector and scalar potential. This allows us to write the Schrodinger Equation for such a particle (mass m and charge q) as:

$$i\hbar \frac{\partial \Psi(\mathbf{r},t)}{\partial t} = \left[\frac{1}{2m} \left(-i\nabla - q\mathbf{A}(\mathbf{r},t) \right)^2 + qV(\mathbf{r},t) \right] \Psi(\mathbf{r},t) \tag{3}$$

where $\mathbf{A}$ and V are the vector and scalar potentials, respectively. Imposing gauge invariance [4],

$$\begin{aligned} \mathbf{A}' &= \mathbf{A} + \nabla\lambda \\ V' &= V - \frac{1}{c}\frac{\partial \lambda}{\partial t} \end{aligned} \tag{4}$$

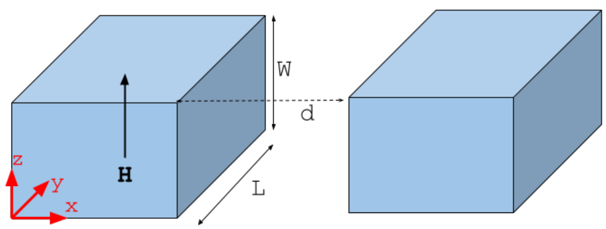

Figure 1: Josephson Junction Diagram

and applying it to the above SE:

$$i\hbar\frac{\partial\Psi'(\mathbf{r},t)}{\partial t} = \left[\frac{1}{2m}\left(-i\hbar\nabla - q\mathbf{A}'(\mathbf{r},t)\right)^2 + qV'(\mathbf{r},t)\right]\Psi'(\mathbf{r},t) \tag{5}$$

These are simultaneously satisfied when the wavefunction acquires a phase shift [2]: $\Psi' = \Psi e^{iq\lambda/c}$, noting $\Psi' = |\Psi'|e^{i\phi'}$, we solve for the phase shift associated with gauge invariance:

$$\phi' = \phi + i\frac{q\lambda}{c} \tag{6}$$

Additionally, from [6], using equations 3 and 5, we can find:

$$\nabla\phi' = \nabla\phi - \frac{q}{c}\mathbf{A} \tag{7}$$

3 Derivations

3.1 1D Josephson Junctions

Figure 1 depicts a schematic of a Josephson junction which is two superconductors separated by a small gap filled with a non-superconducting meteral. As mentioned earlier, the wavefunction of superconductors can be written as $\Psi_i = \sqrt{n_i}e^{i\phi_i}$. For the Josephson junction seen in figure 1 each superconductor can be assigned a wavefunction Ψ_1 and Ψ_2. The insulating gap is small, so the two wavefunctions remain weakly coupled with the coupling constant K. We begin with the Schrodinger Equation with a coupling constant [1]:

$$i\frac{\partial\Psi_1(r)}{\partial t} = E_1\Psi_1 - K\Psi_2$$

Taking the complex conjugate:

$$[-i\frac{\partial\Psi_1^*}{\partial t} = E\Psi_1^* - K\Psi_2^*]$$

The Cooper pair density n_1 is defined as $\Psi_1^*\Psi_1$. Taking the time derivative of this leads to the following:

$$\begin{aligned}\frac{dn_1}{dt} &= \Psi_1^*\frac{d\Psi_1}{dt} + \frac{d\Psi_1^*}{dt}\Psi_1 \\ &= \Psi_1^*\frac{1}{i}(E\Psi_1 - K\Psi_2) - (\frac{1}{i})(E\Psi_1^* - K\Psi_2^*)\Psi_1 \\ &= \frac{K}{i}(-\Psi_2^*\Psi_1 + \Psi_1^*\Psi_2)\end{aligned} \tag{8}$$

However, a time-changing Cooper pair density constitutes an electric current. Furthermore, we assume that the two densities are the same; $n_1 = n_2 = n$, leading to:

$$\frac{dn_1}{dt} = I = \frac{K}{i}n\sin(\phi_2 - \phi_1) = -\frac{dn_2}{dt} \tag{9}$$

We can see from equation 9 that it is possible to sustain a current with no change in voltage. In other words, we have superfluid current across the Josephson Junction.

To continue finding the current of the Josephson Junction, we must consider the gauge invariance property discussed in section 2.2. Integrating over space, we get:

$$\begin{aligned}\int_{x_1}^{x_2}(\nabla_x\phi')\,dx &= \int_{x_1}^{x_2}\left(\nabla_x\phi - \frac{2e}{c}A_x\right)dx \\ &= \phi_2 - \phi_1 - \int_{x_1}^{x_2}\frac{2e}{c}A_x\,dx\end{aligned} \tag{10}$$

Therefore, equation 9 should also include a term with the vector potential, $\mathbf{A}$, to ensure gauge invariance.

$$I = I_c\sin(\phi_2 - \phi_1 + \frac{2e}{c}\int_{x_1}^{x_2}A_x dx) \tag{11}$$

Simplifying further by choosing the temporal gauge [3] where the scalar potential is 0, leading to:

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} = -\frac{\partial\mathbf{A}}{\partial t} \tag{12}$$

$$\int_{x_1}^{x_2}A_x dx = -\int_{x_1}^{x_2}E_x t dx = tV_{12} \tag{13}$$

Where t is the time and V_{12} is the voltage applied across the junction. Substituting into equation 11 we get the main Josephson Junction equations.

$$I = \sin(\phi_2 - \phi_1 + \frac{2e}{c} t V_{12}) \tag{14}$$

From this equation, we can see that the Josephson Junction causes a constant voltage to give rise to an alternating output current. This is called the AC Josephson effect.

3.2 3D Josephson Junctions with applied magnetic field

To introduce a magnetic field, we must look at Josephson Junctions in 3D space. The Josephson Junction can be seen in figure 1 with a magnetic field going in the z direction. The magnetic field can be written as $\mathbf{H} = \nabla \times \mathbf{A}$. Since we know that $\mathbf{H}$ is along the z-axis: $H_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$. From the the geometry of the setup, we only care about A_x and so $H_z = \frac{\partial A_x}{\partial y}$ which makes $A_x = H_z y$. Taking another integral in x: $\int A_x dx = H_z y d$.

Now, substituting equation 11:

$$\begin{aligned} j(y) &= j_c \sin\left(\phi_2 - \phi_1 - \frac{2e}{c} \int A_x \, \mathrm{dx}\right) \hat{x} \\ &= j_c \sin\left(\phi_2 - \phi_1 - \frac{2e}{c} H_z y d\right) \hat{x} \end{aligned} \tag{15}$$

We can then find the current density by integrating over the transverse directions to find the cross section:

$$J = \frac{1}{LW} \int_0^L \mathrm{dy} \int_0^W \mathrm{dz}[j_c \sin(\phi_2 - \phi_1 - \frac{2e}{c} H_z y d)] \tag{16}$$

$$J = \frac{j_c c}{2e H_z d L}[\cos(\phi_2 - \phi_1) - cos(\phi_2 - \phi_1 - \frac{2e}{c} H_z d L)] \tag{17}$$

Note use of the trig identity $\cos(\alpha) - \cos(\beta) = -2\sin(\frac{\alpha-\beta}{2})\sin(\frac{\alpha+\beta}{2})$ and the definition of the flux quanta $\Phi_0 = \frac{ch}{2e} = \frac{c}{4\pi e}$. We also define the magnetic flux of the junction which is $\Phi = \oint \mathrm{dA}(H_z) = H_z L d$. This allows us to simplify the above equation:

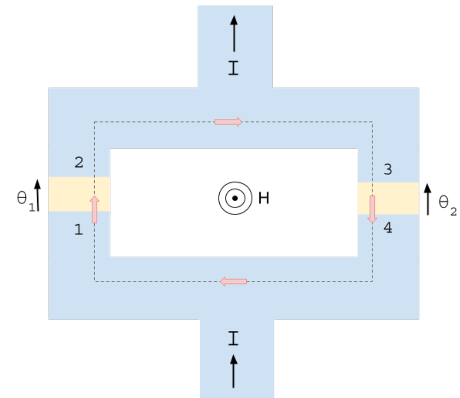

Figure 2: SQUID Diagram

$$J = j_c \frac{\Phi_0}{n\Phi}[\sin(\phi_2 - \phi_1 - \frac{n\Phi}{\Phi_0})\sin(\frac{n\Phi}{\Phi_0})] \tag{18}$$

Since the first *sin* function oscillates between -1 and 1, we can determine the maximum current to be the value of the prefactor:

$$J_{max} = j_c \left| \frac{\Phi_0}{n\Phi} \sin(\frac{n\Phi}{\Phi_0}) \right| \tag{19}$$

As seen above, maximum current J_{max} varies with the applied magnetic flux Φ. The net current arises from the sum of contributions from different parts of the junction which can interfere, constructively or destructively, depending on the phase difference caused by the magnetic field. At certain values of Φ, there is perfect destructive interference, resulting in no net current.

3.3 SQUIDS

Now consider the SQUID device as depicted in 2. The device is formed by connecting two Josephson Junctions in parallel. As detailed in the figure, we must account for the phase of the wavefunction at four locations; either side of both junctions. For convenience, let θ_1 and θ_2 be the jump in phase across each junction. We can use equation 14 to find the total output current as the sum of the two inputs:

$$I = I_{c1} sin(\theta_1) + I_{c2} sin(\theta_2)$$

The bulk of the superconductor has a superfluid current ($\bar{v}_s$) of 0, so [5]:

$$0 = \bar{v_s} = \nabla\theta_i - \frac{2e}{c}\mathbf{A} \implies \nabla\theta_i = \frac{2e}{c}\mathbf{A} \tag{20}$$

We integrate this current over the dashed contour, which is a closed loop integral, so the phase must be some multiple of 2π over the whole integral.

$$2\pi m = \int_1^2 \nabla\theta \mathrm{dl} + \int_2^3 \nabla\theta \mathrm{dl} + \int_3^4 \nabla\theta \mathrm{dl} + \int_4^1 \nabla\theta \mathrm{dl}$$

where m is an integer.

We recognize that the integrals from $1 \to 2$ and $3 \to 4$ become θ_1 and $-\theta_2$, respectively. The remaining integrals from $2 \to 3$ and $4 \to 1$ are essentially the whole loop assuming the junctions are small compared to the size of the loop. This is a reasonable assumption since the junctions made of insulating materials are typically on the Angstrom scale. Therefore, the integrand $\nabla\theta$ can be changed to $\frac{2e}{c}\mathbf{A}$ via equation 20 over the whole contour; this is just the flux passing through the interior of the SQUID.

$$2\pi m = \theta_1 - \theta_2 + \frac{2e}{c}\Phi$$

Recalling $\Phi_0 = \frac{ch}{2e}$,

$$\theta_1 - \theta_2 = 2\pi m - 2\pi\frac{\Phi}{\Phi_0}$$

We can change into generalized coordinate system θ:

$$\theta_1 = \theta - \frac{\pi\Phi}{\Phi_0} + 2\pi m \tag{21}$$

$$\theta_2 = \theta + \frac{\pi\Phi}{\Phi_0} \tag{22}$$

The current splits as it enters the SQUID, meaning that both sides current should be equal.

$$I_{c1} = I_{c2} = \frac{I}{2} \equiv I_c$$

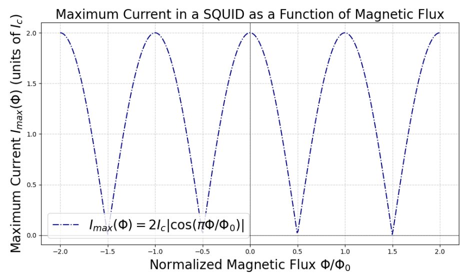

Figure 3: SQUID Max Current

Returning to the AC Josephson Junction expression, equation 14:

$$I = I_c[\sin(\theta - \frac{\pi\Phi}{\Phi_0}) + \sin(\theta + \frac{\pi\Phi}{\Phi_0})] = 2I_c \sin\theta \cos(\frac{\pi\Phi}{\Phi_0})$$

Extracting the prefactor when the $sin\theta$ term takes its maximum value of 1, we find the maximum current to be:

$$I_{\max}(\Phi) = 2I_c |\cos(\frac{\pi\Phi}{\Phi_0})| \tag{23}$$

This can be further seen in figure 3.

The critical current vanishes at $\Phi = (n + \frac{1}{2})\Phi_0$. Since Φ depends on the flux through the SQUID loop and Φ_0 is tiny, it is possible to measure very small magnetic fields. We can calculate the theoretical precision of a representative device of area 1 cm^2:

$$B = \frac{0.5\Phi_0}{A} = 1.035 * 10^{-7} \text{ Gauss} = 1.035 * 10^{-11} \text{ T}$$

4 Discussion

The extreme sensitivity of the SQUID is exploited in many applications to find small changes in magnetic fields. The measurement is performed by recording the oscillations in output current when magnetic flux varies. The measurement is equivalent to moving along the horizontal axis of figure 3 while recording oscillations in output current.

5 Conclusions

We show how a thin insulating junction between two superconducting metals comes to exhibit AC and

DC Josephson Effects using basic statements from Ginzburg-Landau superconductivity. We then show how these equations can be used to form a SQUID device, in which two Josephson junctions are placed in parallel in a small superconducting loop, to measure small changes in magnetic flux. These are fundamentally the same phenomena leveraged by newer SQUIDs which use high-temperature superconductors and have an even greater degree of precision.

Appendix

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References

[1] Aleksei Alekseevich Abrikosov. *Fundamentals of the Theory of Metals.* Courier Dover Publications, 2017.

[2] Jim Branson. Physics 130a quantum mechanics notes. Online lecture notes, 2003. Posted on April 2, 2003.

[3] Michael Creutz. Quantum electrodynamics in the temporal gauge. *Annals of Physics*, 117(2):471–483, 1979.

[4] David J. Griffiths. *Introduction to Electrodynamics.* Cambridge, London, fourth edition, 2017.

[5] Yuki Sato, Aditya Joshi, and Richard Packard. Direct measurement of quantum phase gradients in superfluid He^4 flow. *Physical Review Letters*, 98(19):195302, May 2007.

[6] Barton Zwiebach. Mit 8.06 quantum physics iii, spring 2018 - lecture 14. Lecture, Spring 2018.